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NON-METABELIAN SOLUBLE GROUPS INVOLVING THE LUCAS NUMBERS

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Abstract

In this paper we investigate the class G(n,k) of groups of small deficiencies. This class is of interest for several reasons. It is relevant to the study of 2-generator 2-relator groups, and it adds to the relatively few examples of the soluble groups of derived length 3. Also, the order of G(n,k), when it is finite, is equal to $n(1+\alpha)(g_n-1-(-1)^n)$ where α=h.c.f.(n,3) and g_n denotes the Lucas numbers defined by $g_o=2, g_1=1, g_{n+2}=g_n+g_{n+1}$, $(n\geq 2)$.

Introduction

The family of infinite classes of finitely presented finite groups which are soluble of derived length 3 is small, for examples see [2,3,5 and 6]. The purpose of this paper is to examine the groups

$G(n,k)=\langle a,b \mid a^2=b^n=1,\ ab^kab^{-1}ab^2ab^{-k}ab^{-2}ab=1\rangle$,

and the related deficiency zero groups

$G(n)=\langle a,b \mid a^2=b^n,\ a^iba^jb^{-1}a^kb^2a^{\ell}b^{-1}a^mb^{-2}a^tb=1\rangle$,

where $i,j,k,\ell,m,t \in \{\pm 1\}$, to show that, among these groups there are certain infinite subclasses of non-metabelian finite soluble groups. The presentations of these groups arise from the investigation of (2,n)-groups $\langle a,b \mid a^2=b^n=1,\ w(a,b)=1\rangle$ which were studied in [4] and [5] in all the cases when $w(a,b) = ab^h ab^i ab^j ab^k$ $h,i,j,k \in \{\pm 1, \pm 2\}$.

The Reidemeister-Schreier algorithm in the form given in [1] will be used to find the presentations of subgroups. The notation used here are standard and are consistent with that of [5]. The notation (m,n) will be used for the highest common factor of the integers m and n; [a,b] for the commutator $a^{-1}b^{-1}ab$; and G' denotes the derived group of the group G.

If G is 2 finite group with presentation G=F/R (F is a free group of finite rank) then Schur multiplier M (G) of G is the subgroup $(F'\cap R)/[F,R]$, where [F,R] is the group generated by all commutators $x^{-1}y^{-1}xy$, $x \in F$, $y \in R$.

A group S is a Schur extension of a group G if there exists a subgroup $A \leq S$ such that $S/A \cong G$ and $A \leq S' \cap Z(S)$ (Z(S) is the centre of S).

A covering group C of a group G is a group which contains a subgroup A which satisfies the conditions $C/A \cong G$ and $A \leq C' \cap Z(C)$.

Keywords: Combinatorial group theory, presentation of groups, Fibonacci sequences

2. The groups G(n,k)

Define the Lucas numbers $g_o=2, g_1=1$ $g_{n+2}=g_n+g_{n+1}$, $(n\geq 2)$, which are related to the Fibonacci numbers $f_0=f_1=1, f_{n+2}=f_n+f_{n+1}$, $(n\geq 2)$, via the relation $g_n=f_{n-1}+f_{n+1}$. Our result in this section is:

Theorem 2.1.

(i) For every integer $n\geq 1$, G(n,1) is a finite soluble group of order $n(1+\alpha)(g_n-1-(-1)^n)$, where $\alpha=(n,3)$;

(ii) If (n,3) =1, G(n,1) is soluble of derived length 3, and is metabelian, otherwise;

(iii) $G(n,k) \cong G(n,1)$ if (n,k) =1, otherwise G(n,k) is infinite. We prove this theorem in some stages, and the following lemma is of some help in the proof.

Lemma 2.2.

(i) If $(n,6)=3$, then $(f_{n-1}, -1+f_{n-2})=2$; and for every $n\geq 0, \sum_{k=0}^{n} f_k = -1+f_{n+2}$;

(ii) If $(n,6)=6$ then, $\sum_{k=1}^{-1+n/6} f_{n-1-6k} = (-2+f_{n-4})/4$, and $\sum_{k=1}^{-1+n/6} f_{n-6k} = (-3+f_{n-3})/4$;

(iii) If $(n,6)=3$ then, $\sum_{k=1}^{(n-3)/6} f_{n-1-6k} = (1/4)f_{n-4}$, and $\sum_{k=1}^{(n-3)/6} f_{n-6k} = (-1+f_{n-3})/4$;

(iv) $f_{n-1}.f_{n+1}-f_n^2=(-1)^{n-1}$, and $g_n=f_n+f_{n-2}(n>1)$.

Proof. We prove (i). The other identities may be proved in a similar way. Let $n=6m+3$. Then, for every integer $k\geq o$ we have

$(f_{n-1}, -1+f_{n-2}) = (f_{6m+2}, -1+f_{6m+1})$

$= \left((-1)^k f_{k+1}+f_{6m-k}, (-1)^{k+1} f_k + f_{6m+1-k}\right)$.

Let $a_k = \left((-1)^k (f_{k+1}+f_{6m-k}, (-1)^{k+1} f_k + f_{6m+1-k}\right)$ and $b=(f_{6m+2}, -1+f_{6m+1})$.Then

$$
\begin{aligned}
a_0 &= (f_1+f_{6m}, -f_0+f_{6m+1}) \\
&= (1+f_{6m}, -1+f_{6m+1}) \\
&= (1+f_{6m}-1+f_{6m+1}, -1+f_{6m+1}) \\
&= (f_{6m}+f_{6m+1}, -1+f_{6m+1}) \\
&= (f_{6m+2}, -1+f_{6m+1}) = b,
\end{aligned}
$$

Now, let $a_k=b$, we have

$$
\begin{aligned}
a_{k+1} &= ((-1)^{k+1} f_{k+2} + f_{6m-k-1}, (-1)^{k+2} f_{k+1} + f_{6m-k}) \\
&= ((-1)^{k+2} f_{k+1} + f_{6m-k}, (-1)^{k+1} f_{k+2} + f_{6m-k-1}) \\
&= ((-1)^{k} f_{k+1} + f_{6m-k}, (-1)^{k+1} f_{k+2} + f_{6m-k-1} + (-1)^k f_{k+1} + f_{6m-k}) \\
&= ((-1)^{k} f_{k+1} + f_{6m-k}, (-1)^{k+1} (f_{k+2} - f_{k+1}) + (f_{6m-k-1} + f_{6m-k})) \\
&= ((-1)^{k} f_{k+1} + f_{6m-k}, (-1)^{k+1} f_k + f_{6m-k+1}) \\
&= a_k = b.
\end{aligned}
$$

Note that we have used the properties $(x,y)=(y,x)$ and $(x,y)=(x,y+x)$. Let $k=3m$ then,

$$
\begin{aligned}
(f_{n-1}, -1+f_{n-2}) &= (f_{3m+1}+f_{3m}, -f_{3m}+f_{3m+1}) \text{ (m odd or even)} \\
&= (f_{3m+2}, f_{3m-1}) \\
&= (2f_{3m}+f_{3m-1}, f_{3m-1}) \\
&= (2f_{3m}, f_{3m-1}) \text{ (for, } (f_{3m}, f_{3m-1})=1) \\
&= (2, f_{3m-1}) = 2 \text{ (for, } f_{3m-1} \text{ is even).}
\end{aligned}
$$

Note that f_i is even if and only if $i\equiv 2$ or $-1 \pmod 6$. The last part of (i) can be proved by induction. ∎

First we consider $G(n,1)$ and show that it is finite for every n. The subgroup $H=\langle b, aba\rangle$ of $G(n,1)$ has index 2 in $G(n,1)$, for, using coset enumeration and defining two cosets $1=H$ and $1a=2$ shows that $2b=2$ and $1b=1$. We shall adopt the standard practice of using i to denote both a coset and a representative of that coset throughout the paper when we use the Reidemeister-Schreier algorithm. Let $x=a$ and $y=aba$, thus, from the subgroup generators we obtain the relations $1.b=x.1$ and $2.b=y.2$ between coset representatives. Now, the relations $i.a^2=i$, $i.b^n=i$ and $i.(abab^{-1}ab^2ab^{-1}ab^{-2}ab)=i$, $(i=1,2)$ yield the only non-trivial relations $x^n=y^n=1$, $y^{-1}x^2y=xy^{-1}x^2$ and $x^{-1}y^2x=yx^{-1}y^2$ for the subgroup H. i. e., $|G(n,1):H|=2$ and we have

$$H=\langle x,y \mid x^n=y^n=1, y^{-1}x^2y=xy^{-1}x^2, x^{-1}y^2x=yx^{-1}y^2\rangle.$$

Let $W=H'$. W can be generated by $\{\omega_i : i=1,\dots,n\}$ where $\omega_i=x^iy^{-1}x^{-i+1}$, for the abelianized group H/H' shows that $y^{-1}x\varepsilon H'$, $|H/H'|=n$ and by defining n cosets $1=\langle\omega_1,\dots,\omega_n\rangle=n$. We now use the above mentioned method to get a presentation for W, and we'll get

$$W=\langle\omega_1,\dots,\omega_n \mid \omega_i\omega_{i+1}=\omega_{i-1}, [\omega_i^2, \omega_{i+1}]=1, \omega_n\omega_{n-1}\dots\omega_2\omega_1=1, (i=1,\dots,n)\rangle,$$

where indices are reduced modulo n. Now, we have:

Lemma 2.3. For every $n\geq 1$, $|W/W'|=g_n-1-(-1)^n$.

Proof. The abelianized relations of W give us that:

$w_n=w_1w_2$, $w_{n-1}=w_1^2w_2$, $w_{n-2}=w_1^3w_2^2,\dots$

And in general $w_{n-i}=w_1^{f_{i+1}}.w_2^{f_i}$, $i=0,1,\dots,n-3$. (This can be proved by induction on i). Substitute for $w_i(i\geq 3)$ in the relations $w_2w_3=w_1$, $w_3w_4=w_2$ and $w_nw_{n-1}\dots w_2w_1=1$, thus, we will get $W/W'=\langle w_1, w_2 \mid r_1=r_2=r_3=[w_1, w_2]=1\rangle$ where $r_1=w_1^{-1+f_{n-2}} w_2^{1+f_{n-3}}$, $r_2=w_1^{f_{n-1}}.w_2^{-1+f_{n-2}}$ and

$$r_3=w_1^{\sum_o^{n-2} f_1} . w_2^{1+\sum_o^{n-3} f_i}.$$

The relation $r_3=1$ is redundant, for, $r_3=w_1^{-1+f_n}.w_2^{f_{n-1}}$ (by 2.2-(i))

$$=w_1^{-1+f_{n-1}+f_{n-2}} \cdot w_2^{(-1+f_{n-2})+(1+f_{n-3})}$$

$$=r_1 r_2=1 \quad (\text{for}, [w_1,w_2]=1).$$

So, $|W/W'| = \det M$ where

$$M=\begin{bmatrix} -1+f_{n-2} & 1+f_{n-3} \\ f_{n-1} & -1+f_{n-2} \end{bmatrix}.$$

Then, $|W/W'| = |f_{n-2}^2-f_{n-1}f_{n-3}+1-2f_{n-2}-f_{n-1}|$

$$= (-1)^{n-1}-1+f_{n-2}+(f_{n-2}+f_{n-1})$$

$$= (-1)^{n-1}-1+(f_{n-2}+f_n) = (-1)^{n-1}-1+g_n$$

(For, $f_{m-1}\ f_{m+1}-f_m^2=(-1)^{m-1}$ and $g_n=f_n+f_{n-2}$, $n\geq 2$). This completes the proof. ❑

Lemma 2.4. $G(n,1)$ is finite; and if $(n,3)=1$ then $|G(n,1)|=2n(g_n-1-(-1)^n)$.

Proof. We showed that $|G/H|$, $|H/W|$ and $|W/W'|$ are finite, so it is sufficient to show that W' is finite.

Consider the central subgroup $K=\langle w_1^2,\ldots,w_n^2\rangle$ of W. Then, a coset enumeration shows that

$$|W:K| = \begin{cases} 1 & ,\text{if } (n,3)=1 \\ 4 & ,\text{if } (n,3)=3. \end{cases}$$

(We may define four cosets as $1=K$, $1w_1=2$, $1w_2=3$ and $3w_1=4$). This proves that Z(W) (the centre of W) is of finite index in W. The result now follows from the well-known theorem due to Schur (see 2.2. of [7], for example).

To complete the proof, let $(n,3)=1$. Then, $|W/K|=1$, i.e., W is an abelian group. So, $|W|=g_n-1-(-1)^n$ follows from 2.3., and then the result is immediate.❑

Now, let us consider the case $(n,3)=3$. Simplifying the presentation of W is substantial, and the following lemma is a key result for this simplification.

Lemma 2.5. In W, for every $k\geq 3$, w_k can be expressed in terms of w_1 and w_2 as follows:

(i) $w_k=w_2^{-f_{k-2}} . w_1^{f_{k-3}}$,if k=3 or 1(mod6)

(ii) $w_k=w_1^{-f_{k-3}} . w_2^{f_{k-2}}$,if $k\equiv\pm 2$ or 0(mod6)

(iii) $w_k=w_1^{f_{k-4}} . w_2^{f_{k-2}} . w_1^{f_{k-5}}$,if $k\equiv -1$ (mod6)

Proof. By induction on k and using the Lemma 2.2.❑

The following two lemmas now give us the order of W.

Lemma 2.6. Let $(n,6)=6$. Then, W can be presented as

$$W=\langle w_1,w_2 \mid r_1=r_2=1,$$
$$(w_1w_2)^2(w_2w_1)^{-2}=1,$$
$$[w_1^2,w_2]=[w_2^2,w_1]=1\rangle$$

where, $r_1=w_1^{1+f_{n-3}} . w_2^{1-f_{n-2}}$ and $r_2=w_1^{1-f_{n-2}} . w_2^{f_{n-1}}$. Moreover, $|W|=2(g_n-2)$.

Proof. Observe that the relations $[w_i^2, w_{i+1}]=1$, $(i>3)$ all are redundant, for, suppose $(i,6)=3$ (the proofs are similar in the other cases), then by the above lemma we conclude that

$$[w_i^2, w_{i+1}]=[w_1^{-f_{i-2}} . w_2^{-f_{i-3}} . w_1^{-f_{i-2}} . w_2^{f_{i-3}} . w_1^{-f_{i-2}} . w_2^{f_{i-1}}]$$
$$=[w_2^{-f_{i-3}} . w_1^{f_{i-2}}]^2 . [w_1^{f_{i-2}} . w_2^{-f_{i-3}}]^{-2}$$

(this is true because $[w_1^2, w_3]=1$ is equivalent to $[w_2^2 w_1]=1$ and since f_{i-1} is even then, $w_2^{f_{i-1}}$ commutes with w_1.)

On the other hand $[w_3^2, w_4]=1$ is equivalent to the relation $(w_1 w_2)^2=(w_2w_1)^2$. Using this relation and the fact that f_{i-3} and f_{i-2} are both odd integers, give us the validity of the relation

$$[w_2^{f_{i-3}} w_1^{f_{i-2}}]^2=[w_1^{f_{i-2}} w_2^{-f_{i-3}}]^2.$$

So $[w_i^2, w_{i+1}]=1$ for every $i>3$.

Substitute w_n and w_{n-1} in the two relations $w_1w_2=w_n$ and $w_nw_1=w_{n-1}$ to get the required results $r_1=r_2=1$. Obviously, $w_2w_3=1$, $w_3w_4=w_2$,..., and $w_{n-1}w_n=w_{n-2}$ yield the trivial or redundant relations.

To complete the proof we show that $w_n w_{n-1} \cdots w_2 w_1=1$ is also redundant. Let $X_0=w_nw_{n-1}w_{n-2}w_{n-3}w_{n-4}w_{n-5}$. Substitute for w_i in terms of w_1 and w_2, then X_0 becomes

$$X_o=w_1^{-f_{n-3}+2f_{n-6}-f_{n-9}} . w_2^{f_{n-2}-2f_{n-5}+f_{n-8}},$$

because, $(n,6)=6$ then f_{n-4} and f_{n-7} are even, and other powers of w_1 and w_2 in the experession of X_O are odd numbers, hence the result follows from the relations $[w_1^2, w_2]=[w_2^2, w_1]=1$.

The properties of Fibonacci numbers, now give us the following identities

$$-f_{n-3}+2f_{n-6}-f_{n-9}=-4f_{n-7}, \text{and } f_{n-2}-2f_{n-5}+f_{n-8}=4f_{n-6},$$

so $x_o=w_1^{-4f_{n-7}} \cdot w_2^{4f_{n-6}}$, and $w_n w_{n-1} \ldots w_2 w_1=1$ becomes

$$w_1^A \cdot w_2^B=1$$

where, $A=-4 \sum_{k=0}^{-2+n/6} f_{n-6k-7}$ and $B=4(1+\sum_{k=0}^{-2+n/6} f_{n-6k-6})$. As a result of 2.2.-(ii), $w_1^A \cdot w_2^B=1$ is equivalent to $r_1 \cdot r_2=1$, and so is redundant.

To find the order of W, consider the subgroup $L=\langle w_1^2, w_2\rangle$ of W where we can easily see that $|W: L|=4$ and using the Reidemeister-Schrier algorithm gives us the following presentation

$$L=\langle x, y \mid [x, y]=1, x^{(1+f_{n-3})/2} \cdot y^{1-f_{n-2}}=1, x^{(1+f_{n-2})/2} \cdot y^{f_{n-1}}=1\rangle.$$

The order of this abelian group equals $(g_n-2)/2$ which may be found by the matrix method as well as Lemma 2.3. Then, $|W|=2(g_n-2)$.□

Lemma 2.7. Let $(n,6)=3$. Then W can be presented as follows:

$$W=\langle w_1, w_2 \mid r_3=r_4=1, (w_1 w_2)^2(w_2 w_1)^{-2}=1, [w_1^2, w_2]=[w_2^2, w_1]=1\rangle$$

where, $r_3=(w_1w_2)^{-1}w_2^{-f_{n-2}} \cdot w_1^{f_{n-3}}$ and $r_4=w_1^{1+f_{n-2}} \cdot w_2^{-f_{n-1}}$. Moreover $|W|=2g_n$.

Proof. In an almost similar way to that of 2.6., using 2.2.- (ii), and considering the subgroup $L=\langle w_1^2, w_2\rangle$ of W which is of index 2 in W in this case.□

Proof of theorem 2.1. (i) comes from 2.3, 2.4, 2.6, and 2.7. To prove (ii) we see that

$$\begin{cases} |H''|=|W'|=1, & \text{if } (n,3)=1 \\ |H''|=|W'|=2, & \text{if } (n,3)=3. \end{cases}$$

(for, $|W/W'|=g_n-1-(-1)^n$). On the other hand H' is a subgroup of G', for, $yx^{-1}=ab^{-1}ab \in G'$, so $w_i \in G'$ for every i. Since

$$|G: H'|=|G: H| \cdot |H: H'|=2n=|G: G'|,$$

then, $G' \cong H'$. Thus, the result follows immediately.

To prove (iii), if $d=(n,k)\neq 1$ we add the relation $b^d=1$ to those of G(n,k) and get the infinite free product $Z_2 * Z_d$ as a homomorphic image of G(n,k), so, G(n,k) is infinite. Now, let $d=(n,3)=1$ and

$$R=abab^{-1}ab^2ab^{-1}ab^{-2}ab, \quad S=ab^kab^{-1}ab^2ab^{-k}ab^{-2}ab.$$

R=1 gives that $aba=b^{-1}ab^2abab^{-2}ab$. Raising both sides to the power k and getting the relation S=1. Conversely, S=1 and $b^n=1$ yield R=1, because, there exist integers β and γ such that

$$\beta n+\gamma k=1.$$

So, S=1 yields $ab^{\gamma k}a=b^{-1}ab^2ab^{-\gamma k}ab^{-2}ab$. Hence, substituting for γk and considering $b^n=1$, gives the result R=1. This completes the proof.□

3. Deficiency zero groups

Consider the

$$G(n)=\langle a,b \mid a^2=b^n, a^i ba^j b^{-1} a^k b^2 a^{\ell} b^{-1}a^m b^{-2} a^t b=1\rangle$$

where, $i,j,k,l,m,t \in \{\pm 1\}$. Let $A=i+j+k+\ell+m+t$. Obviously, if A=0 then G(n) is an infinite group (for, it is a group with positive deficiency). And if A=2 or 4 or 6 we'll get the following three non-isomorphic groups:

$$G_1=\langle a,b \mid a^2=b^n, abab^{-1} ab^2 ab^{-1} ab^{-2} ab=b^{2n}\rangle,$$

$$G_2=\langle a,b \mid a^2=b^n=abab^{-1}ab^2ab^{-1} ab^{-2}ab\rangle,$$

$$G_3=\langle a,b \mid a^2=b^n, abab^{-1} ab^2 ab^{-1} ab^{-2} ab=1\rangle$$

respectively. In this section, Our results concerning the finite groups involving Lucas numbers, are the following two theorems.

Theorem 3.1. G_1 is a finite soluble group.

Proof. The subgroup $\langle a^2\rangle$ of G_1 is a central subgroup and a^2 belongs to G_1 (one may consider G_1/G_1'). Also, $G/\langle a^2\rangle \cong G(n,1)$. So, G_1 is a Schur extension of G(n,1). This means that G_1 is a homomorphic image of a covering group of G(n,1). Since G(n,1) is finite (Section 2), thus, G_1 is finite.□

Theorem 3.2. For every $n\equiv\pm 1$ (mod6), G_3 is a finite group of order $6ng_n$.

Proof. Consider the subgroup $K=\langle b, aba^{-1}\rangle$ of G_3.

Define two costs 1=K and la=2 to show that $|G_3:K|=2$. A similar method as in Section 2 may be used to find a presentation for K. Thus, we'll get

$$K=\langle x,y \mid x^n=y^n, (yx^{-1}y^2x^{-1}y^{-2}x)x^{3n}=1, (xy^{-1}x^2y^{-1}x^{-2}y)x^{3n}=1\rangle.$$

We show that the relation $x^{3n^2}=1$ holds in K. The second relation of K may be rewritten as

$$xyx^{-1}=x^{-3n}(y^2xy^{-2}) \quad (\text{for, } \langle x^n\rangle \text{ is central}).$$

Raising both sides to the power n:

$$xy^nx^{-1} = x^{-3n^2}(y^2x^ny^{-2}).$$

So, $x^{3n^2}=1$ (Considering the relation $x^n=y^n$).

Now, we prove that x^n has period 3. Suppose, m is the least positive integer such that $x^{mn}=1$ holds in K and consider

$$K/K' = \langle x,y \mid x^n = y^n,\ yx^{3n-1}=1,\ x=y^{-3n+1},\ [x,y]=1\rangle$$
$$= \langle x \mid x^{3n^2}=1,\ x^{6n}=1\rangle$$

which is isomorphic to $\mathbb{Z}_{3n}$ or $\mathbb{Z}_{6n}$ if n is odd, or if n is even, respectively. Let $n = 6q\pm1$. If $(m,3)=1$ we have $K/K' \cong Z_n$ which is a contradiction, then, $(m,3)=3$. Let $m=3q'$, say. Since $(m,n) = (3q', 6q\pm1)=1$ and m divides 3n, hence, m divides 3, i.e. m=3. Thus, $\langle x^n\rangle$ has order 3. We add the relation $x^n=1$ to those of K and get the group

$$\langle x,y \mid x^n= y^n=1,\ yx^{-1}y^2x^{-1}y^{-2}x=1,\ xy^{-1}x^2y^{-1}x^{-2}y=1\rangle$$

which is a factor group of K by a central subgroup of order 3. However, this group considered to have order $2ng_n$ (Section 2), consequently, $|G_3| = 6ng_n$. ❑

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